PMM U.S.S.R.,Vol.45,pp.861-864 0021-8928/82/6 0861 $7.50/0

UDC 531.36

CARNOT—OSTROGRADSKII THEOREMS FOR THE SYSTEMS WITH NONSTATIONARY CONSTRAINTS*

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Variation in the kinetic energy of mechanical systems with nonstationary constaints moving impulsively under the conditions of the Carnot–Ostrogradskii theorems /1/, is studied. Theorems concerning the change in kinetic energy of a constrained system under the action of impulsive, nonretentive constraints, are proved, and an example is solved.

Let a motion of a mechanical system be restricted by perfect holonomic and linear nonholonomic constraints. The position of the system is given by the generalized coordinates $q_1, \ldots, q_r$ ($r \geqslant m$, m denoting the number of degrees of freedom), i.e. the generalized velocities satisfy, at any instant of time, the equations

$$\dot{q}_{m+p} - \sum_{j=1}^{m} b_{pj} q_j^{\cdot} - b_p = 0 \quad (p = 1, \ldots, r-m) \tag{1}$$

where the coefficients b_{pj} and b_p are continuously differentiable functions of the generalized coordinates and time. At a certain instant additional perfect nonretentive constraints of the form

$$q_{l+\nu}^{\cdot} = \sum_{i=1}^{l} a_{\nu i} q_i^{\cdot} \quad (\nu = 1, \ldots, m-l) \tag{2}$$

are imposed on the system. The coefficients $a_{\nu i}$ satisfy the same demands as the coefficients of (1).

In what follows, we shall use the terminology of /2/, according to which the constraints of the type (2) shall be called catastatic (all terms of the equation containing generalized velocities), and those of type (1) shall be called acatastatic. Let us separate the expression for the kinetic energy θ of the system obtained by eliminating the dependent velocities with help of the constraint equations (1), into a group θ_2 of terms of second power with respect to the generalized velocities, group θ_1 of terms linear in the generalized velocities, and group θ_0 of terms indpendent of the generalized velocities

$$\theta = \theta_2 + \theta_1 + \theta_0$$

Constructing the equations of motion of the system with constraints (1) in the usual manner and integrating them over the time interval of the impulsive interaction with the constraints (2), we obtain, under the assumption that active impulsive forces are absent, the equations of impulsive motion /3/

$$\left(\frac{\partial\theta_2}{\partial q_i^{\cdot}} + \sum_{\nu=1}^{m-l} a_{\nu i}\frac{\partial\theta_2}{\partial \dot{q}_{l+\nu}}\right)\Bigg|_{-}^{+} = 0 \quad (i = 1, \ldots, l) \tag{3}$$

($f\,|_-^+ = f^+ - f^-$ the indices plus and minus refer to the characteristics corresponding to the state of the system before and after the impact). Function θ_2 can be regarded as the kinetic energy of a mechanical system which we shall call the reduced system. The impact resulting from the application of the constraints will take place if the rates of deformation of the constraints (2) (α_ν) at the initial instant are negative

$$q_{l+\nu}^{\cdot -} - \sum_{i=1}^{l} a_{\nu i} q_i^{\cdot -} = \alpha_\nu \quad (\nu = 1, \ldots, m-l) \tag{4}$$

In the case of an elastic collision the post-impact state is characterized by the positive rates of relaxation of the constraints (β_ν) (4)

$$q_{l+\nu}^{\cdot +} - \sum_{i=1}^{l} a_{\nu i} q_i^{\cdot +} = \beta_\nu \quad (\nu = 1, \ldots, m-l) \tag{5}$$

*Prikl.Matem.Mekhan.,45,No.6,1138-1142,1981

The process of collision can be conveniently represented in the form of two phases. The end of the first phase is determined by the instant at which the deformation of the constraints ceases and their relaxation (restoration) begins. The generalized velocities $(q_1^{\cdot *}, \ldots, q_m^{\cdot *})$ corresponding to this state must satisfy the constraint equations (2).

Let us assume that the deformation of all the constraints is terminated simultaneously. The Carnot—Ostrogradskii theorems refer to the changes in the kinetic energy occurring during the first phase of the impulsive action of the constraints, during the second phase of the collision, and in the course of the whole collision. Proofs of the theorems are known /1/ for the restricted systems with holonomic stationary constraints. To prove the first theorem we multiply each of the equations of impusive motion (3) for the first phase by $q_i^{\cdot *}$ and sum it over i. This yields

$$\sum_{i=1}^{l}\left(\frac{\partial\theta_2}{\partial q_i^{\cdot}}\right)\Big|_-^* q_i^{\cdot *}+\sum_{\nu=1}^{m-l}\left(\frac{\partial\theta_2}{\partial q_{l+\nu}^{\cdot}}\right)\Big|_-^*\sum_{i=1}^{l}a_{\nu i}q_i^{\cdot *}=0 \tag{6}$$

Taking into account the fact that the generalized velocities at the end of the first phase satisfy the equations (2), we reduce (6) to the form

$$\sum_{j=1}^{m}\frac{\partial\theta_2^*}{\partial q_j^{\cdot *}}q_j^{\cdot *}-\sum_{j=1}^{m}\frac{\partial\theta_2^-}{\partial q_j^{\cdot -}}q_j^{\cdot *}=0 \tag{7}$$

Since θ_2 is a homogeneous quadratic form, (7) yields the following relation

$$2\theta_2^*-2\theta_2(q^{\cdot -}, q^{\cdot *})=0 \quad \left(\theta_2(q^{\cdot -}, q^{\cdot *})=\frac{1}{2}\sum_{k,\, j=1}^{m}m_{kj}q_k^{\cdot -}q_j^{\cdot *}\right)$$

where $\theta_2(q^{\cdot -}, q^{\cdot *})$ is a bilinear form of the generalized velocities (m_{kj} are the inertial coefficients). The well known properties of the bilinear forms allows the following substitution:

$$\theta_2(q^{\cdot -}, q^{\cdot *})=\tfrac{1}{2}[\theta_2(q^{\cdot -}, q^{\cdot -})+\theta_2(q^{\cdot *}, q^{\cdot *})-\theta_2(q^{\cdot -}-q^{\cdot *}, q^{\cdot -}-q^{\cdot *})] \tag{8}$$
$$\theta_2(q^{\cdot -}, q^{\cdot -})=\theta_2^-, \quad \theta_2(q^{\cdot *}, q^{\cdot *})=\theta_2^*$$

after which we obtain

$$\theta_2^- - \theta_2^* = \theta_2(q^{\cdot -}-q^{\cdot *},\ q^{\cdot -}-q^{\cdot *}) \tag{9}$$

We obviously arrive at the same result when the constraints imposed are maintained after the impact, since in this case the whole process of collision consists of a single phase (perfectly inelastic impact). The relation (9) corresponds to the following assertion of the theorem: under the impulsive motion of the system caused by the application of the perfect permanent catastatic constraints, the loss of kinetic energy in the reduced system is equal to the kinetic energy of the generalized velocities lost by the reduced system. The second theorem is obtained by transforming, in the same manner, the equations of impuslive motion (3) constructed for the second phase of the collision. As a result, we arrive at the relation

$$\theta_2^+ - \theta_2^* = \theta_2(q^{\cdot +}-q^{\cdot *},\ q^{\cdot +}-q^{\cdot *}) \tag{10}$$

which expresses the second theorem: during the second phase of the collision in which the system is released from the perfect catastatic constraints, the kinetic energy gained by the reduced system is equal to the kinetic energy of the generalized velocities acquired by the reduced system.

In proving the third theorem we shall assume that the elastic properties of the interaction between the system and the nonretentive constraints are characterized by the equal values of the ratios of the rates at which the constraints diminish, to the deformation rates, i.e.

$$\beta_\nu=-\mu\alpha_\nu \quad (\nu=1,\ldots,m-l) \tag{11}$$

The physical sense of the coefficient μ can be established using the results of /5/, namely, the coefficient μ coincides with the coefficient of restitution. In other words, we assume that the coefficients of restitution are the same in the case of elastic interaction between the material points of the system and the nonretentive constraints. This condition enables us to close, with the help of (4) and (5), the equations (3) with respect to the generalized velocities $q_1^{\cdot+}, \ldots, q_m^{\cdot+}$

$$\left(q_{l+\nu}^{\cdot+} - \sum_{i=1}^{l} a_{\nu i} q_i^{\cdot+}\right) = -\mu\left(q_{l+\nu}^{\cdot-} - \sum_{i=1}^{l} a_{\nu i} q_i^{\cdot-}\right) \quad (\nu = 1, \ldots, m-l)$$

Multiplying equations (3) by the factors $(q_i^{\cdot+} + \mu q_i^{\cdot-})$ respectively and summing over the index $i\,(i = 1, \ldots, l)$, we obtain

$$\sum_{i=1}^{l}\left(\frac{\partial\theta_2}{\partial q_i^{\cdot}}\right)\Big|_-^+ (q_i^{\cdot+} + \mu q_i^{\cdot-}) + \sum_{\nu=1}^{m-l}\left(\frac{\partial\theta_2}{\partial q_{l+\nu}^{\cdot}}\right)\Big|_-^+ \sum_{i=1}^{l} a_{\nu i}(q_i^{\cdot+} + \mu q_i^{\cdot-}) = 0 \tag{12}$$

Making in equality (12) the substitution

$$\sum_{i=1}^{l} a_{\nu i} q_i^{\cdot-} = q_{l+\nu}^{\cdot-} - \alpha_\nu, \quad \sum_{i=1}^{l} a_{\nu i} q_i^{\cdot+} = q_{l+\nu}^{\cdot+} - \beta_\nu \quad (\nu = 1, \ldots, m-l)$$

and taking into account the conditions (3) and relations

$$\sum_{i=1}^{m}\left(\frac{\partial\theta_2}{\partial q_i^{\cdot}}\right)^+ q_i^{\cdot-} = \sum_{i=1}^{m}\left(\frac{\partial\theta_2}{\partial q_i^{\cdot}}\right)^- q^{\cdot+} = 2\theta_2(q^{\cdot-}, q^{\cdot+})$$

we find, that (12) assumes the form

$$2\theta_2^+ - 2\mu\theta_2^- - 2(\mu - 1)\theta_2(q^{\cdot-}, q^{\cdot-}) = 0 \tag{13}$$

Substituting in (13) the bilinear form (8) and carrying out the obvious manipulations, we obtain

$$\theta_2^- - \theta_2^+ = \frac{1-\mu}{1+\mu}\theta_2(q^{\cdot-} - q^{\cdot+}, q^{\cdot-} - q^{\cdot+}) \tag{14}$$

Thus we have proved the third theorem, namely that the loss of kinetic energy in the reduced system due to its interaction with the perfect catastatic constraints (with the ratios of the deformation rates to the constraint relaxation rates equal to each other) is equal to $(1-\mu)/(1+\mu)$, and represents the part of the kinetic energy of the reduced system computed for the generalized velocity losses.

Example. A homogeneous sphere of radius r and unit mass rolls without slipping on a rough horizontal plate rotating, together with a smooth vertical wall, about the OZ-axis (see Fig.1), with angular velocity Ω. At some instant the sphere hits the wall, and the coefficient of restitution of the impact is $\varkappa$. We require to set up the equations for determining the post-impact velocities.

Let us choose a fixed $OXYZ$ coordinate system as shown in Fig.1. The kinetic energy of the sphere is given by the equation

$$2T = x^{\cdot 2} + y^{\cdot 2} + z^{\cdot 2} + \rho^2(\omega_x^2 + \omega_y^2 + \omega_z^2)$$

where ρ is the radius of inertia of the sphere relative to any diameter, and x, y, z are the coordinates of the center. We write the projections of the angular velocity of the sphere $\omega_x, \omega_y, \omega_z$ in terms of the Euler angles φ, ψ, ϑ

$$\omega_x = \vartheta^{\cdot}\cos\psi + \varphi^{\cdot}\sin\vartheta\sin\psi, \quad \omega_z = \varphi^{\cdot}\cos\vartheta + \psi^{\cdot}, \quad \omega_y = \vartheta^{\cdot}\sin\psi - \varphi^{\cdot}\sin\vartheta\cos\psi \tag{15}$$

and substitute these equations into the expression for the kinetic energy

$$2T = x^{\cdot 2} + y^{\cdot 2} + z^{\cdot 2} + \rho^2(\vartheta^{\cdot 2} + \varphi^{\cdot 2} + \psi^{\cdot 2} + 2\varphi^{\cdot}\psi^{\cdot}\cos\vartheta)$$

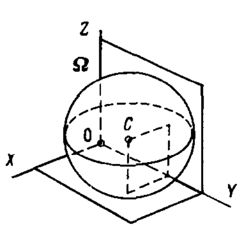

Fig.1

The absence of slipping corresponds to the constant action exerted by the constraints

$$x^{\cdot} - r\omega_y + \Omega y = 0, \quad y^{\cdot} + r\omega_x - \Omega x = 0, \quad z^{\cdot} = 0 \tag{16}$$

Choosing $\vartheta^{\cdot}, \varphi^{\cdot}, \psi^{\cdot}$ as the independent generalized velocities, we obtain the following expression for the kinetic energy of the reduced system:

$$2\theta_2 = (r^2 + \rho^2)\vartheta^{\cdot 2} + (r^2\sin^2\vartheta + \rho^2)\varphi^{\cdot 2} + \rho^2(\psi^{\cdot 2} + 2\varphi^{\cdot}\psi^{\cdot}\cos\vartheta) \tag{17}$$

At the instant of impact, the following additional constraint appears:

$$x^{\cdot} = -\Omega y$$

which, according to (15) and (16), is expressed in terms of the generalized coordinates as follows:

$$\vartheta^{\cdot} \sin\psi - \varphi^{\cdot} \sin\vartheta \cos\psi = 0$$

Using this expression together with (17), we construct the equations (3)

$$(r^2 \sin^2\vartheta + \rho^2)\, \Delta\varphi^{\cdot} \sin\psi + (r^2 + \rho^2)\, \Delta v^{\cdot} \sin\vartheta \cos\psi + \rho^2 \Delta\psi^{\cdot} \cos\vartheta \sin\psi = 0,\ \Delta\psi^{\cdot} + \Delta\varphi^{\cdot} \cos\vartheta = 0\ (\Delta q^{\cdot} = q^{\cdot +} - q^{\cdot -})$$

The third equation is obtained using a generalization of the Carnot—Ostrogradskii theorem (14) and taking into account the relation $\mu = \varkappa$

$$\theta_2^{+} - \theta_2^{-} = -\frac{1-\varkappa}{1+\varkappa}\, \theta_2 (\Delta\vartheta^{\cdot},\ \Delta\varphi^{\cdot},\ \Delta\psi^{\cdot})$$

or more simply

$$\vartheta^{\cdot +} \sin\psi - \varphi^{\cdot +} \sin\vartheta \cos\psi = -\varkappa\, (\vartheta^{\cdot -} \sin\psi - \varphi^{\cdot -} \sin\vartheta \cos\psi)$$

The resulting equations of impulsive motion of a sphere coincide, as expected, with the impact equations obtained in /3/ (Ch. 111, Sect.9, Ex.5) where a motion along a fixed surface was considered.

In conclusion we shall show how to generalize the theorems to the case of elastic interaction with acatastatic constraints

$$q_{l+\nu}^{\cdot} = \sum_{i=1}^{l} a_{\nu i} q_i^{\cdot} + a_\nu \quad (\nu = 1, \ldots, m - l)$$

The only difference in the proof will consist of the fact that all arguments will be repeated for the relative velocities. The relative velocities of some state of the system are defined as the differences between the corresponding generalized velocities and the translational generalized velocities, the latter represented by any set of the generalized velocities satisfying the acatastatic constraints. If $q_1^{\cdot e}, \ldots, q_m^{\cdot e}$ are the translational veclocities, then the relative velocities before, after, and at the end of the first phase of collision, are equal to the differences

$$(q_j^{\cdot -} - q_j^{\cdot e}),\quad (q_j^{\cdot +} - q_j^{\cdot e}),\quad (q_j^{\cdot *} - q_j^{\cdot e}) \quad (j = 1, \ldots, m)$$

In this case the kinetic energy θ_2 in the equations (9), (10) and (14) expressing the generalizations of the Carnot—Ostrogradskii theorems, will also be calculated for the relative velocities.

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Translated by L.K.